ON SYMMETRIC PRESSURE OF A CIRCULAR STAMP ON AN ELASTIC HALF-SPACE IN THE PRESENCE OF ADHESION

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The problem of symmetric pressure of a plane circular stamp on an elastic half-space in the presence of adhesion between the stamp and support was considered in the paper by Mossakovskii [1]. Using the method of integral transforms, Mossakovskii reduces the solution of the problem to the mixed plane problem in potential theory for two analytic functions, which is then solved by reduction to a single linear problem.

The more general problem of a rigid stamp, circular in plane of stamp in the elastic half-space in the presence of adhesion, was considered in the work of Ufliand [2 and 3]. The problem here was solved in toroidal coordinates with the help of the Mehler-Fok integral transforms.

In the present work we shall for simplicity, consider the axially symmetric problem of pressure of a rigid cross-section stamp of circular on an elastic half-space in the presence of adhesion. The problem is solved using a cylindrical coordinate system. The biharmonic stress functions of A. Love are sought in the form of Hankel integrals. The determination of arbitrary functions of integration is reduced to a system of two dual integral equations containing Bessel functions of the first kind.

By application of Fourier transforms the solution of this system is reduced to Privalov's boundary value problem and is expressed with the help of quadratures. The solution of the problem of contact of a circular stamp with adhesion but without axial symmetry may be obtained in an analogous way.

1. Formulation of problem. We consider the problem of a circular stamp on an elastic half-space $r \geqslant 0$ in the presence of adhesion between the stamp and the foundation. We assume that on the circular region beneath the stamp the normal and transverse displacement is given. We assume that the remaining part of the surface of the half-space is, for simplicity, free of all external forces.

For the solution of the problem we use a cylindrical coordinate system.

Boundary conditions for the problem will have the form

$$\left.\begin{aligned}\sigma_z(r,0)&=0\\ \tau_{rz}(r,0)&=0\end{aligned}\right\}\quad (r>a),\qquad \left.\begin{aligned}u_r(r,0)&=f_1(r)\\ u_z(r,0)&=f_2(r)\end{aligned}\right\}\quad (r<a) \tag{1.1}$$

Here $f_1(r)$ and $f_2(r)$ are smooth functions defining the form of the surface of the stamp and the character of the contact.

Apart from Equation (1.1) we also make use of the conditions of axial symmetry

$$u_r(0,z)=0,\qquad \tau_{rz}(0,z)=0\qquad (0\leqslant z<\infty) \tag{1.2}$$

We also assume that at infinity, points belonging to the half-space are stress- and strain-free.

The biharmonic function of Love for this problem is taken in the form of Hankel integrals

$$\varphi(r,z)=\int_0^\infty \lambda\,[A(\lambda)+zB(\lambda)]\,e^{-\lambda z}J_0(\lambda r)\,d\lambda \tag{1.3}$$

Here $J_i(x)$ are Bessel functions of the first kind with real argument; functions $A(\lambda)$ and $B(\lambda)$ are subject to determination from the boundary conditions.

Making use of the usual equations expressing stresses and displacements by means of the function $\varphi(r, z)$, we will have

$$\begin{aligned}\sigma_z(r,z)&=\int_0^\infty[(1-2\nu)B+\lambda A+\lambda zB]\,\lambda^3e^{-\lambda z}J_0(\lambda r)\,d\lambda\\ \tau_{rz}(r,z)&=\int_0^\infty(\lambda A-2\nu B+\lambda zB)\,\lambda^3e^{-\lambda z}J_1(\lambda r)\,d\lambda\\ u_z(r,z)&=-\frac{1}{2\mu}\int_0^\infty[2(1-2\nu)B+\lambda A+\lambda zB]\,\lambda^2e^{-\lambda z}J_0(\lambda r)\,d\lambda\\ u_r(r,z)&=-\frac{1}{2\mu}\int_0^\infty(\lambda A-B+\lambda zB)\,\lambda^2e^{-\lambda z}J_1(\lambda r)\,d\lambda\end{aligned} \tag{1.4}$$

where μ is the shear modulus and ν is the Poisson's ratio.

Expression (1.4) identically satisfies condition (1.2).

Meeting conditions (1.1) for the determination of the unknown functions $A(\lambda)$ and $B(\lambda)$ we obtain a system of two dual integral equations

$$\int_0^\infty \lambda\,(A^*+B^*)\,J_0(\lambda r)\,d\lambda=0\qquad (r>a) \tag{1.5}$$

$$\int_0^\infty (A^* + B^*) J_0(\lambda r)\, d\lambda = F_1(r) \qquad (r < a)$$
$$\int_0^\infty \lambda (A^* - \alpha B^*) J_1(\lambda r)\, d\lambda = 0 \qquad (r > a) \tag{1.6}$$
$$\int_0^\infty (A^* - \alpha B^*) J_1(\lambda r)\, d\lambda = F_2(r) \qquad (r < a)$$

Here we use the following notation

$$(1 - 2\nu)\lambda^2 B(\lambda) = B^*(\lambda), \qquad \lambda^3 A(\lambda) = A^*(\lambda), \qquad \alpha = \frac{2\nu}{1 - 2\nu} \tag{1.7}$$

$$F_1(r) = -2\mu f_1(r) - \int_0^\infty B^*(\lambda) J_0(\lambda r)\, d\lambda$$
$$F_2(r) = -2\mu f_2(r) + \int_0^\infty B^*(\lambda) J_1(\lambda r)\, d\lambda \tag{1.8}$$

2. Solution of system of dual integral equations (1.5) and (1.6). Dual integral equation of the form (1.5) and (1.6) have been investigated by Titchmarsh [4], Sneddon [5 and 6], Akhiezer [7], Noble [8], Burlak [9] and others.

The solution of these equations may be represented in the form

$$A^*(\lambda) + B^*(\lambda) = \frac{2}{\pi}\int_0^a \cos\lambda t\, dt \frac{d}{dt}\int_0^t \frac{rF_1(r)\, dr}{(t^2 - r^2)^{1/2}} \tag{2.1}$$

$$A^*(\lambda) - \alpha B^*(\lambda) = \frac{2}{\pi}\int_0^a \sin\lambda t\, dt \frac{d}{dt}\left[t\int_0^t \frac{F_2(r)\, dr}{(t^2 - r^2)^{1/2}}\right] \tag{2.2}$$

Subtracting the second equation from the first we get

$$(1 + \alpha) B^*(\lambda) = \frac{2}{\pi}\int_0^a \cos\lambda t\, dt \frac{d}{dt}\int_0^t \frac{rF_1(r)\, dr}{(t^2 - r^2)^{1/2}} - \frac{2}{\pi}\int_0^a \sin\lambda t\, dt \frac{d}{dt}\left[t\int_0^t \frac{F_2(r)\, dr}{(t^2 - r^2)^{1/2}}\right] \tag{2.3}$$

Substituting the expression (1.8) into the latter, results in

$$B^*(\lambda) = -\frac{4\mu(1 - 2\nu)}{\pi}\left\{\int_0^a \cos\lambda t\, dt \frac{d}{dt}\int_0^t \frac{rf_1(r)\, dr}{(t^2 - r^2)^{1/2}} - \right.$$
$$\left. - \int_0^a \sin\lambda t\, dt \frac{d}{dt}\left[t\int_0^t \frac{f_2(r)\, dr}{(t^2 - r^2)^{1/2}}\right]\right\} - \tag{2.4}$$
$$-\frac{2(1 - 2\nu)}{\pi}\left\{\int_0^a \cos\lambda t\, dt \frac{d}{dt}\int_0^t \frac{r\, dr}{(t^2 - r^2)^{1/2}}\int_0^\infty B^*(\lambda) J_0(\lambda r)\, d\lambda + \right.$$
$$\left. + \int_0^a \sin\lambda t\, dt \frac{d}{dt}\left[t\int_0^t \frac{dr}{(t^2 - r^2)^{1/2}}\int_0^\infty B^*(\lambda) J_1(\lambda r)\, d\lambda\right]\right\}$$

Here the relation $1 + \alpha = (1 - 2\nu)^{-1}$ is also used.

Changing the order of integration in the integral of the second term in the right-hand side of expression (2.4) and making use of the identities

$$\frac{d}{dt}\int_0^t \frac{rJ_0(\xi r)\,dr}{(t^2 - r^2)^{1/2}} = \cos \xi t, \qquad \frac{d}{dt}\left[t\int_0^t \frac{J_1(\xi r)\,dr}{(t^2 - r^2)^{1/2}}\right] = \sin \xi t \tag{2.5}$$

we obtain from (2.4), in order to determine $B^*(\lambda)$, the integral equation

$$B^*(\lambda) = \int_0^\infty K(\lambda - \xi)\, B^*(\xi)\, d\xi + \varphi(\lambda) \tag{2.6}$$

Here

$$K(\lambda - \xi) = M\,\frac{\sin a(\lambda - \xi)}{\lambda - \xi}, \qquad M = -\frac{2(1 - 2\nu)}{\pi} \tag{2.7}$$

$$\varphi(\lambda) = -\frac{4\mu(1 - 2\nu)}{\pi}\left\{\int_0^a \cos \lambda t\, dt\, \frac{d}{dt}\int_0^t \frac{rf_1(r)\,dr}{(t^2 - r^2)^{1/2}} - \int_0^a \sin \lambda t\, dt\, \frac{d}{dt}\left[t\int_0^t \frac{f_2(r)\,dr}{(t^2 - r^2)^{1/2}}\right]\right\} \tag{2.8}$$

Titchmarsh [4], Rapoport [10], Gakhov [11] and others have concerned themselves with the solution of equations of the form (2.6). The case of the system of such equations was considered by Krein and Gokhberg [12]. In their work, the integral equation under consideration was reduced to the problem of Privalov. In the solution of Equation (2.6) we shall use the results obtained by Rapoport.

Since the kernel of integral equation (2.7) is an even function, then the transform of the Fourier kernel (2.7) is a real function, on account of which the index of integral equation (2.6) $n = 0$. At the same time, the integral equation has a unique solution in class L^2 for any right-hand side $\varphi(\lambda) \in L^2$.

Let us assume that $B^*(\lambda) = \varphi(\lambda) = 0$ if $-\infty < \lambda < 0$ and introduce the notation

$$F(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} B^*(\lambda)\, e^{ix\lambda}\, d\lambda, \qquad G(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \varphi(\lambda)\, e^{ix\lambda}\, d\lambda \tag{2.9}$$

Then, from the solution of the corresponding Privalov problem it is easy to obtain, for the function $F(x)$, the expression

$$F(x) = G(x)\,[1 - \chi(x)] + \frac{1 - \sqrt{3 - 4\nu}}{2\pi i W(x)}\left|\frac{x - a}{x + a}\right|^{i\alpha}\int_{-a}^{a} \frac{G(t)}{t - x}\left(\frac{a + t}{a - t}\right)^{i\alpha} dt \tag{2.10}$$

where

$$\chi(x) = \begin{cases} \gamma & |x| < a, \\ 0 & |x| > a, \end{cases} \qquad W(x) = \begin{cases} 3 - 4\nu, & |x| < a \\ 1 & |x| > a \end{cases}$$
$$\alpha = \frac{\beta}{2\pi}, \qquad \beta = \ln(3 - 4\nu), \qquad \gamma = \frac{5 - 8\nu - \sqrt{3 - 4\nu}}{2(3 - 4\nu)} \tag{2.11}$$

and the integral in equation (2.10) is taken as its principal value in Cauchy's sense.

Making use now of the inverse Fourier transform, from (2.9) to (2.11) we obtain, for the unknown function $B^*(x)$,

$$B^*(x) = \varphi(x) - \frac{\gamma}{\sqrt{2\pi}} \int_{-a}^{a} G(t) e^{-ixt} dt + \frac{1 - \sqrt{3 - 4\nu}}{2\pi i \sqrt{2\pi}} \left[\frac{\pi i}{\sqrt{3 - 4\nu}} \int_{-a}^{a} G(t) e^{-ixt} dt + \right.$$
$$\left. + \frac{1 - \sqrt{3 - 4\nu}}{3 - 4\nu} \int_{-a}^{a} G(y) \left(\frac{a + y}{a - y} \right)^{i\alpha} dy \int_{-a}^{a} \left(\frac{a - t}{a + t} \right)^{i\alpha} \frac{e^{-ixt}}{t - y} dt \right]$$

This expression may be reduced to the form

$$B^*(x) = \varphi(x) - \frac{4 - 6\nu - \sqrt{3 - 4\nu}}{\pi(3 - 4\nu)} \int_{0}^{\infty} \frac{\sin a(x - y)}{x - y} \varphi(y) dy +$$
$$+ \frac{2(1 - \nu) - \sqrt{3 - 4\nu}}{2\pi^2(3 - 4\nu)} \int_{0}^{\infty} F(x, y) \varphi(y) dy \tag{2.12}$$

where we introduce the notation

$$F(x, y) = -i \int_{-a}^{a} \int_{-a}^{a} \left[\frac{(a + z)(a - t)}{(a - z)(a + t)} \right]^{i\alpha} \frac{e^{i(yz - xt)}}{t - z} dz\, dt =$$
$$= \int_{-a}^{a} \int_{-a}^{a} \sin\left[\frac{\beta}{2\pi} \ln \frac{(a - z)(a - t)}{(a + z)(a + t)} - (tx + yz) \right] \frac{dz\, dt}{t + z} \tag{2.13}$$

Here also the values of the integrals in the right-hand side are taken as their principal values.

In the special case when $\nu = 0.5$ or $a = 0$, from (2.12) we obtain

$$B^*(x) = \varphi(x)$$

On the other hand, if $a \to \infty$, then on basis of formula [4]

$$\lim_{a \to \infty} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin a(x - y)}{x - y} \varphi(y) dy = \varphi(x), \qquad \lim_{a \to \infty} F(x, y) = 2\pi^2 \delta(y - x)$$

and (2.12) yields

$$B^*(x) = \frac{\varphi(x)}{3 - 4\nu}$$

The unknown function $A^*(\lambda)$ is determined from Equation (2.1) with the help of (2.12). Finally, functions $A(\lambda)$ and $B(\lambda)$ are determined by the relations (1.7).

Making use of the obtained values for $A(\lambda)$ and $B(\lambda)$ we may, with the help of quadrature, determine the stress arising underneath the stamp.

The system of two dual integral equation of the form (1.5) and (1.6) was also considered in a recent paper by Szefer [13]. However, in investigating this system the author allowed some inaccuracies and reduced the solution to a system of two integro-differential equations, the solution of which is not given.

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